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Spatial stochastic resonance in one-dimensional Ising systems

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The one-dimensional Ising model is analytically studied in a spatially periodic and oscillatory external magnetic field using the transfer-matrix method. For low enough magnetic field intensities the correlation between the external magnetic field and the response in magnetization presents a maximum for a given temperature. The phenomenon can be interpreted as a resonance phenomenon induced by the stochastic heat bath. This ''spatial stochastic resonance'' is realized in the equilibrium state and not as a dynamical response to the external time-periodic driving. [S1063-651X(99)50210-9]

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I. INTRODUCTION

Many recent papers [1–9] revealed unequivocally the phenomenon of stochastic resonance (SR) [10] in the kinetic Ising model driven by a temporary oscillating magnetic field. SR was anticipated by considering the Ising model as a system of coupled two-state oscillators in the stochastic forcefield of thermal fluctuations. In this sense the system has all the ingredients necessary to observe the classical phenomenon of stochastic resonance.

In the present paper we intend to study the one-dimensional ferromagnetic Ising model in a spatially periodic and oscillatory $B(i)$ magnetic field. We consider $\langle B(i)\rangle_i=0$ (the brackets denote a special averaging) and $B(i+\lambda)=B(i)$. 2λ is the spatial period of the magnetic field, i the lattice points, $i=1,2,3,\dots,2p\lambda$, and $p=1,2,\dots$ an integer.

The Hamiltonian of the system is written as

$$H=-J\sum_{i=1}^{2p\lambda} S(i)S(i+1)-\mu\sum_{i=1}^{2p\lambda} B(i)S(i), \qquad (1)$$

with μ the magnetic moment of the $S(i)=\pm 1$ Ising spins. We impose periodic boundary conditions, thus $S(2p\lambda+1)=S(1)$. The magnetic field is taken stationary in time.

Due to the oscillatory nature of the magnetic field at the $T=0$ thermodynamic temperature (and not to high magnetic field intensities) the $\sigma=\langle B(i)S(i)\rangle$ correlation is greatly reduced. (The brackets in the correlation denotes both a spatial and ensemble average.) This is simply understandable realizing that the infinite correlation length $[\xi(0)=\infty]$ competes with the finite 2λ period of $B(i)$. At $T=\infty$ the leading stochastic contribution gives $\sigma=0$. We expect that for a given finite temperature the ξ correlation length will be of the order of the λ period of $B(i)$ and thus the σ correlation will reach a maximal value. This spatial resonancelike phenomenon is induced by the stochastic force field (temperature) for the energetically frustrated system at $T=0$. Spatial SR type effects have been already reported in one-dimensional chains of coupled SR elements driven by time-periodic signals. The noise enhanced spatiotemporal synchronization was numerically demonstrated and discussed for a chain of linearly coupled bistable elements [11]. Experimental evidence of such behavior was obtained in an array of coupled diode resonators [12]. The main difference between these earlier studies and the present one is that our Hamiltonian (1) is now time independent. The expected resonancelike phenomenon is realized in the equilibrium state and not as a dynamical response to external time-periodic driving.

II. METHOD

To give an exact solution for the proposed problem we choose the most simplest possible $B(i)$ configuration with the above imposed properties. We choose $B(i)=B$ for $i=2n\lambda+1$ $(n=0,1,2\dots p-1)$, $B(i)=-B$ for $i=(2n+1)\lambda+1$, and $B(i)=0$ for all other lattice points. We are interested in the $\langle S(1)\rangle$ average magnetization at the $i=1$ position from where the σ correlation is easily determined. From the chosen magnetization profile we get

$$\sigma=\langle B(i)S(i)\rangle=pB(\langle S(1)\rangle-\langle S(\lambda+1)\rangle). \qquad (2)$$

From symmetry arguments $\langle S(1)\rangle=-\langle S(\lambda+1)\rangle$, and we can write

$$\sigma_p=2pB\langle S(1)\rangle. \qquad (3)$$

In order to determine $\langle S(1)\rangle$ we calculate (i) the $Z_{2p\lambda}$ partition function of the system, (ii) the $Z_{2p\lambda}^{+}$ partition function for $S(1)=1$ imposed condition, and (iii) and the $Z_{2p\lambda}^{-}$ partition function for the $S(1)=-1$ imposed condition. We get the desired $\langle S(1)\rangle$ value, as

$$\langle S(1)\rangle=\frac{Z_{2p\lambda}^{+}-Z_{2p\lambda}^{-}}{Z_{2p\lambda}}. \qquad (4)$$

During our calculations we use several matrices whose explicit forms are given in the Appendix.

III. $L=2\lambda$ LENGTH CHAIN ($p=1$)

With the notations $j=J/kT$ and $h=\mu B/kT$ (T is the temperature of the system, and k is the Boltzmann constant) the partition function $Z_{2\lambda}$ is written as

$$Z_{2\lambda}=\sum_{S(1)}\sum_{S(2)}\ldots\sum_{S(2\lambda)}\exp\left\{j\left[\sum_{i=1}^{2\lambda}S(i)S(i+1)\right]+hS(1)-hS(\lambda+1)\right\} \tag{5}$$

(the sums are for $S(i)=\pm 1$).

We use the transfer-matrix method to calculate $Z_{2\lambda}$,

$$\begin{aligned}Z_{2\lambda}=\sum_{S(1)}\sum_{S(2)}\ldots\sum_{S(2\lambda)}&\langle S(1)|I_+|S(2)\rangle\\ &\times\langle S(2)|I_0|S(3)\rangle\cdots\langle S(\lambda)|I_0|S(\lambda+1)\rangle\\ &\times\langle S(\lambda+1)|I_-|S(\lambda+2)\rangle\\ &\times\langle S(\lambda+2)|I_0|S(\lambda+3)\rangle\cdots\langle S(2\lambda-1|I_0|S(2\lambda)\rangle\\ &\times\langle S(2\lambda)|I_0|S(1)\rangle\end{aligned}$$

with the I_0 and $I_\pm$ matrices given in the Appendix [see Eq. (A1)]. It is now immediately apparent that

$$Z_{2\lambda}=\mathrm{Tr}(I_+I_0^{\lambda-1}I_-I_0^{\lambda-1}) \tag{6}$$

[where Tr(A) denotes the trace of the A matrix]. Introducing the M [see Eq. (A2)] diagonal matrix we get

$$I_+=MI_0, \tag{7}$$

$$I_-=M^{-1}I_0, \tag{8}$$

$$Z_{2\lambda}=\mathrm{Tr}(W), \tag{9}$$

$$W=MI_0^{\lambda}M^{-1}I_0^{\lambda}. \tag{10}$$

Choosing a representation where I_0 becomes diagonal we get

$$I_0'=UI_0U^{-1}, \tag{11}$$

$$M'=UMU^{-1}, \tag{12}$$

$$W'=M'I_0'^{\lambda}M'^{-1}I_0'^{\lambda}, \tag{13}$$

$$Z_{2\lambda}=\mathrm{Tr}(W') \tag{14}$$

(the U, I_0', and M' matrices are also given in the Appendix). After some elementary algebra one will find

$$\begin{aligned}Z_{2\lambda}=2^{2\lambda}\{&\cosh^2(h)[\cosh^{2\lambda}(j)+\sinh^{2\lambda}(j)]\\ &-2\sinh^2(h)\sinh^{\lambda}(j)\cosh^{\lambda}(j)]\}.\end{aligned} \tag{15}$$

For $Z_{2\lambda}^+$ and $Z_{2\lambda}^-$

$$Z_{2\lambda}^{\pm}=\sum_{S(2)}\sum_{S(3)}\ldots\sum_{S(2\lambda)}\exp\left\{j\left[\sum_{i=1}^{2\lambda}S(i)S(i+1)\right]+hS(1)-hS(\lambda+1)\right\} \tag{16}$$

[for the $\pm$ cases we have $S(1)=\pm 1$, respectively] we perform a similar calculation,

$$\begin{aligned}Z_{2\lambda}^{\pm}=\sum_{S(2)}\sum_{S(3)}\ldots\sum_{S(2\lambda)}&\langle S(2\lambda)|S_\pm|S(2)\rangle\\ &\times\langle S(2)|I_0|S(3)\rangle\cdots\langle S(\lambda)|I_0|S(\lambda+1)\rangle\\ &\times\langle S(\lambda+1)|I_-|S(\lambda+2)\rangle\\ &\times\langle S(\lambda+3)|I_0|S(\lambda+4)\rangle\cdots\langle S(2\lambda-1)|I_0|S(2\lambda)\rangle,\\ Z_{2\lambda}^{\pm}=\mathrm{Tr}(&S_\pm I_0^{\lambda-1}I_-I_0^{\lambda-2}).\end{aligned} \tag{17}$$

The $S_\pm$ matrices are also given in the Appendix [see Eq. (A5)]. Using Eq. (8) we can write

$$Z_{2\lambda}^{\pm}=\mathrm{Tr}(P_\pm I_0^{\lambda}M^{-1}I_0^{\lambda}), \tag{18}$$

$$P_\pm=I_0^{-1}S_\pm I_0^{-1}. \tag{19}$$

Again, we calculate the trace in the representation where I_0 is diagonal and we get

$$\begin{aligned}Z_{2\lambda}^{\pm}=2^{2\lambda-1}e^{\pm h}\{&\cosh(h)[\cosh^{2\lambda}(j)+\sinh^{2\lambda}(j)]\\ &\mp 2\sinh(h)\sinh^{\lambda}(j)\cosh^{\lambda}(j)\}.\end{aligned} \tag{20}$$

With the obtained $Z_{2\lambda}$ and $Z_{2\lambda}^{\pm}$ values $\langle S(1)\rangle$ is easily determined [see Eq. (4)] and for the σ_p [see Eq. (3)] correlation we get finally,

$$\sigma_1=2B\frac{\tanh(2h)}{1+\frac{1}{\cosh(2h)}\left(\frac{1+\tanh^{\lambda}(j)}{1-\tanh^{\lambda}(j)}\right)^2}. \tag{21}$$

IV. $L=2p\lambda$ ($p>1$) LENGTH CHAIN

In order to be able to make the calculations easily in the $p>1$ case we first write $Z_{2\lambda}^{\pm}$ in a more convenient form,

$$Z_{2\lambda}^{\pm}=\mathrm{Tr}(P_\pm' I_0'^{\lambda}M'^{-1}I_0'^{\lambda})=\mathrm{Tr}(R_\pm' W'), \tag{22}$$

$$R_\pm'=P_\pm' M'^{-1}. \tag{23}$$

Is easy to realize that

$$Z_{2\lambda}=Z_{2\lambda}^{+}+Z_{2\lambda}^{-}=\mathrm{Tr}(W'), \tag{24}$$

$$Z_{2\lambda}^{+}-Z_{2\lambda}^{-}=\mathrm{Tr}(R'W'). \tag{25}$$

For the $p>1$ case it is immediate that

$$Z_{2p\lambda}=\mathrm{Tr}(W'^{p}). \tag{26}$$

Writing up the effective forms of $Z_{2p\lambda}^{\pm}$ like in the $p=1$ case, one can also show that

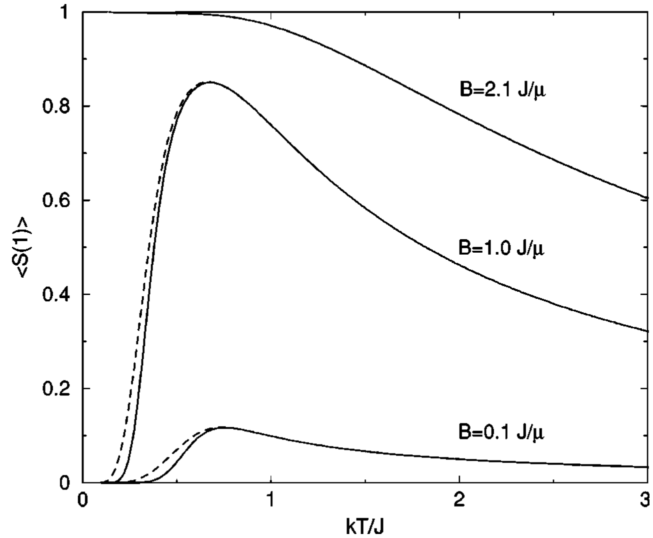

FIG. 1. Characteristic shape of $\langle S(1)\rangle(T)$ for three different magnetic field intensities ($\mu B/k=0.1,1.0,2.1$). The continuous lines are for $p=1$, the dashed ones for $p=\infty$, $\lambda=20$ (lattice spacing) for all curves.

$$Z_{2p\lambda}^{\pm}=\mathrm{Tr}(R'_{\pm}W'^{p}). \tag{27}$$

In the representation where W' is diagonal it is easy now to calculate $\langle S(1)\rangle$ and σ_p. Denoting by χ_1 and χ_2 ($\chi_1 \geqslant \chi_2$) the eigenvalues of W', after some simple algebra we find

$$\sigma_p=2pB\frac{\mathrm{Tr}(R'W'^{p})}{\mathrm{Tr}(W'^{p})}=\sigma_1\frac{\mathrm{Tr}(W')}{\sqrt{\Delta}}\frac{\chi_1^p-\chi_2^p}{\chi_1^p+\chi_2^p}, \tag{28}$$

$$\Delta=\mathrm{Tr}(W')^2-4\det(W'), \tag{29}$$

$$\chi_{1,2}=\frac{\mathrm{Tr}(W')\pm\sqrt{\Delta}}{2}. \tag{30}$$

In the limit $p\rightarrow\infty$ we get the simple formula

$$\sigma_\infty=\sigma_1\frac{\mathrm{Tr}(W')}{\sqrt{(\Delta}}, \tag{31}$$

which is easily computable from the W' matrix given in the Appendix [see Eq. (A6)].

V. DISCUSSION

Equations (21) and (31) give us the $\sigma=\langle B(i)S(i)\rangle$ correlations for the $p=1$ and $p=\infty$ periodic chains. The $\sigma(T)$ correlation is proportional to the $\langle S(1)\rangle(T)$ curves [see Eq. (3)]. In Fig. 1 we plotted $\langle S(1)\rangle(T)$ for three different applied magnetic field intensities.

As expected, for $\mu B/J<2$ (when the interaction with the external field is weaker than the interaction with the neighboring spins) a clear resonancelike behavior is obtained. Both for $p=1$ and $p=\infty$, $\langle S(1)\rangle(T)$ exhibits a clear maximum at a $T_r\neq 0$ resonance temperature. It is also observable that the $\langle S(1)\rangle(T)$ curves for $p=1$ and $p=\infty$ are very close, thus the $p=1$ result is qualitatively well describing the $p>1$ cases as well. The T_r resonance temperature depends both on the B intensity of the applied magnetic field and the characteristic λ distance of the spatial oscillations of $B(i)$. In Fig. 2 we illustrate the $T_r(B)$ dependence, and in Fig. 3 the $T_r(\lambda)$ trend. From Fig. 2 we learn that in the $B\rightarrow 0$ limit the T_r values are converging to a constant (which is dependent on p), and in the $\mu B/J\geqslant 2$ limit $T_r=0$, thus no resonance behavior is obtained. The $T_r(\lambda)$ variations (Fig. 3) are also the ones expected from our phenomenological considerations. In the limit $\lambda\rightarrow\infty$ we get $T_r\rightarrow 0$, and T_r is monotonically decreasing with increasing λ values. It is interesting to note that for the minimal possible λ value ($\lambda=1$) the resonancelike behavior is still present but the $p=1$ and $p=\infty$ curves are much more distant compared to the large λ values case.

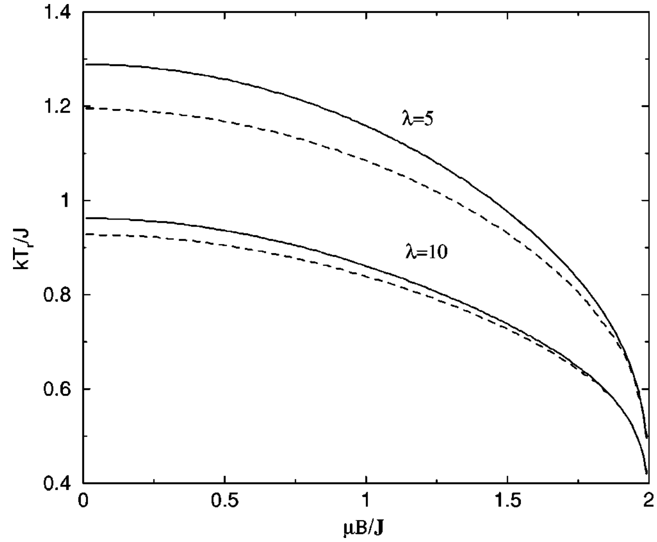

FIG. 2. T_r resonance temperature as a function of the applied magnetic field intensity. We draw the results for two different λ values (λ in units of lattice spacing). The continuous lines are for $p=1$, the dashed ones for $p=\infty$.

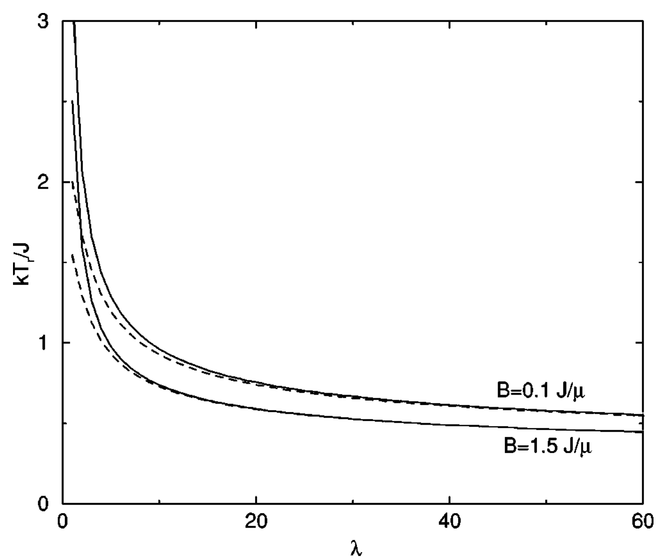

FIG. 3. T_r resonance temperature as a function of the λ length (in units of lattice spacing). We draw the results for two different applied magnetic field intensities. The continuous lines are for $p=1$, the dashed ones for $p=\infty$.

VI. CONCLUSIONS

In the present paper we studied the response of a one-dimensional Ising chain to spatially periodic and oscillatory magnetic fields. Considering the most simple magnetic field profile we exactly solved the problem by using the transfer-matrix method. We found that the $\sigma=\langle B(i)S(i)\rangle$ correlation between the applied magnetic field and the local magnetization exhibits a maximum for a given T_r resonance temperature (Fig. 1). The T_r resonance temperature depends monotonically on the λ spatial oscillation length of the magnetic field (Fig. 3). The value of T_r depends also on the B intensity of the magnetic field, and becomes independent of B in the small B values limit (Fig. 2). For large λ values, the length of the chain ($L=p\times 2\lambda$) has no major influence on the observed resonancelike behavior. The obtained spatial type SR is induced by the stochastic heat bath. It is realized in the equilibrium state and not as a dynamical response to external time-periodic driving.

APPENDIX

$$I_0=\begin{bmatrix} e^{j} & e^{-j} \\ e^{-j} & e^{j}\end{bmatrix},\quad I_\pm=\begin{bmatrix} e^{j\pm h} & e^{-j\pm h} \\ e^{-j\mp h} & e^{j\mp h}\end{bmatrix}, \tag{A1}$$

$$M=\begin{bmatrix} e^{h} & 0 \\ 0 & e^{-h}\end{bmatrix},\quad U=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}, \tag{A2}$$

$$M'=\begin{bmatrix} \cosh(h) & \sinh(h) \\ \sinh(h) & \cosh(h)\end{bmatrix}, \tag{A3}$$

$$M'^{-1}=\begin{bmatrix} \cosh(h) & -\sinh(h) \\ -\sinh(h) & \cosh(h)\end{bmatrix}, \tag{A4}$$

$$I_0'=2\begin{bmatrix} \cosh(j) & 0 \\ 0 & \sinh(j)\end{bmatrix},\quad S_\pm=e^{\pm h}\begin{bmatrix} e^{\pm 2j} & 1 \\ 1 & e^{\mp 2j}\end{bmatrix}, \tag{A5}$$

$$W'=2^{2\lambda}\begin{bmatrix} w'_{11} & w'_{12} \\ w'_{21} & w'_{22}\end{bmatrix}, \tag{A6}$$

$$w'_{11}=\cosh^2(h)\cosh^{2\lambda}(j)-\sinh^2(h)\cosh^{\lambda}(j)\sinh^{\lambda}(j)$$

$$w'_{12}=\sinh(h)\cosh(h)\sinh^{\lambda}(j)[\sinh^{\lambda}(j)-\cosh^{\lambda}(j)$$

$$w'_{21}=\sinh(h)\cosh(h)\cosh^{\lambda}(j)[\cosh^{\lambda}(j)-\sinh^{\lambda}(j)]$$

$$w'_{22}=\cosh^2(h)\sinh^{2\lambda}(j)-\sinh^2(h)\cosh^{\lambda}(j)\sinh^{\lambda}(j)$$

$$P'_\pm=\frac{e^{\pm h}}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1\end{bmatrix}, \tag{A7}$$

$$R'=\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix},\quad R'_\pm=\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1\end{bmatrix}. \tag{A8}$$

[1] Z. Néda, Phys. Rev. E **51**, 5315 (1995).
[2] Z. Néda, Phys. Lett. A **210**, 125 (1996).
[3] K.-T. Leung and Z. Neda, Phys. Lett. A **246**, 505 (1998).
[4] K.-T. Leung and Z. Neda, Phys. Rev. E **59**, 2730 (1999).
[5] J. Javier Brey and A. Prados, Phys. Lett. A **216**, 240 (1996).
[6] L. Schimansky-Geier and U. Siewert, in *Stochastic Dynamics*, edited by L. Schimansky-Geier and T. Pöschel, Lecture Notes in Physics Vol. 484 (Spinger, Berlin, 1997), p. 245.
[7] U. Siewert and L. Schimansky-Geier, Phys. Rev. E **58**, 2843 (1998).
[8] S. W. Sides, R. A. Ramos, P. A. Rikvold, and M. A. Novotny, J. Appl. Phys. **81**, 5597 (1997).
[9] S. W. Sides, P. A. Rikvold, and M. A. Novotny, Phys. Rev. Lett. **81**, 834 (1998).
[10] For review, see, e.g., L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[11] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995); Phys. Rev. E **53**, 2081 (1996).
[12] M. Locher, G. A. Johnson, and E. R. Hunt, Phys. Rev. Lett. **77**, 4698 (1996).